FH

Name: ______________

Class: 12MTZ1

Teacher: MRS HAY

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2011 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page and must show your name and class.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used. Standard Integral Tables are provided.
- Write your name and class in the space provided at the top of this question paper.
- Your solutions will be collected in one bundle stapled in the top left corner. Please arrange them in order, Q1 to 8. The exam paper must be handed in with your solutions.

QUESTION 1 **MARKS**

(a) Find $\int \frac{\sec^2 x}{\sqrt{1+2\tan x}}\, dx$ 2

(b) Use the substitution $u = \tan^{-1} x$ to evaluate 2

$$\int_1^{\sqrt{3}} \frac{1}{(1+x^2)\tan^{-1} x}\, dx$$

(c) Find $\int \tan^{-1} x\ dx$ 3

(d) Let $t = \tan\frac{\theta}{2}$

(i) Show that $d\theta = \frac{2}{1+t^2}\, dt$ 1

(ii) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate 3

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\operatorname{cosec}^2\theta \tan\frac{\theta}{2}\, d\theta$$

(e) Find $\int \frac{x+4}{(x-1)(x^2+4)}\, dx$ 4

QUESTION 2 **BEGIN A NEW PAGE** **MARKS**

(a) Let $z = 1 - 2i$. Express in the form $x + iy$

(i) $\frac{1}{z}$ 1

(ii) $\bar{z}\,(z - \bar{z})$ 1

(iii) $\frac{z}{i} + i\bar{z}$ 2

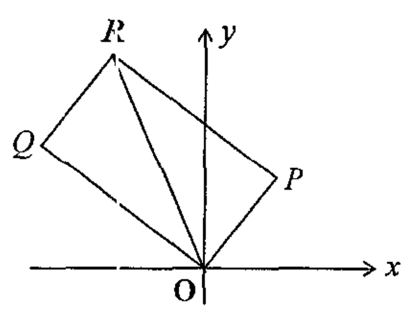
(b) (i) Express $z = 1 - \sqrt{3}\,i$ in modus-argument form. 1

(ii) Show that z^6 is an integer. 2

(c) If $\arg z_1 \neq \arg z_2$, show that $|z_1| + |z_2| > |z_1 + z_2|$. 2

(d) $\text{Arg}(z + 3 - 2i) = \frac{3\pi}{4}$. Sketch the locus of the point P representing z in the Argand diagram and write down its Cartesian equation. 2

(e) Let $z = \cos\alpha + i\sin\alpha$, where α is an angle in the first quadrant.
On the Argand diagram the point P represents z, the point Q represents $i\sqrt{3}\,z$ and the point R represent $z + i\sqrt{3}\,z$.



(i) Explain why $OPRQ$ is a rectangle. 1

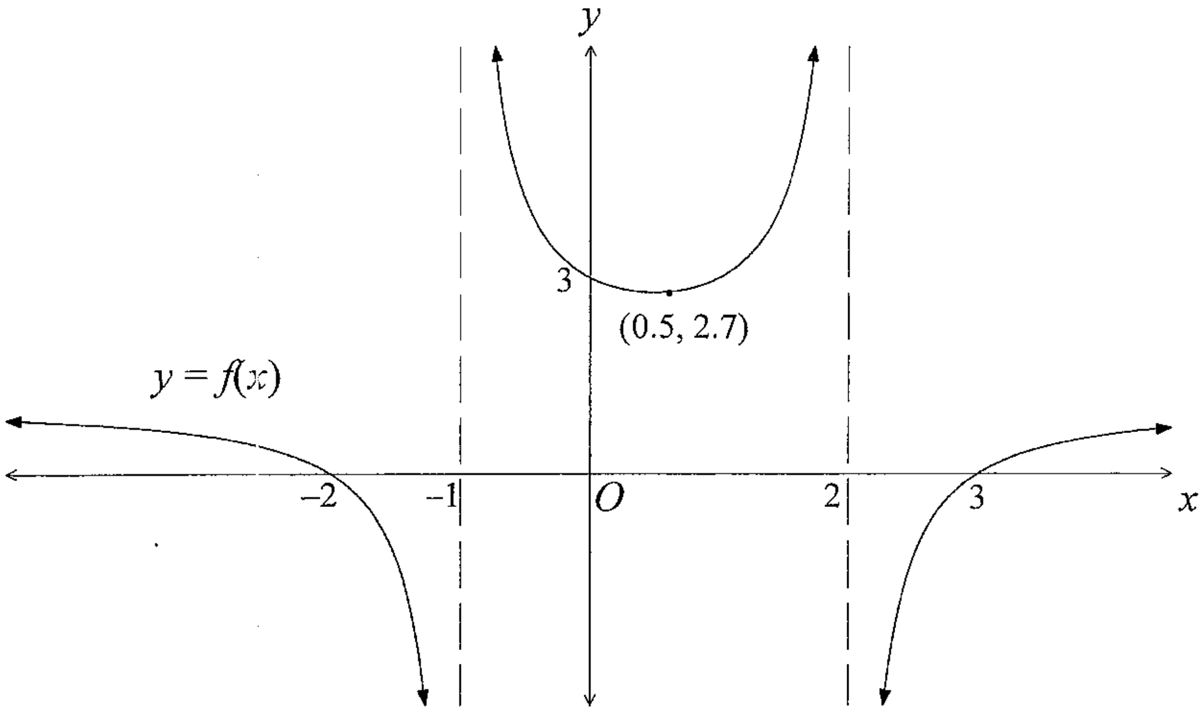
(ii) Show that $|z + i\sqrt{3}\,z| = 2$. 1

(iii) Show that $\text{Arg}\,(z + i\sqrt{3}\,z) = \alpha + \frac{\pi}{3}$. 1

(iv) By considering the imaginary part of $z + i\sqrt{3}\,z$, deduce that 1
$\sin\alpha + \sqrt{3}\cos\alpha = 2\sin\left(\alpha + \frac{\pi}{3}\right)$.

QUESTION 3 **BEGIN A NEW PAGE** **MARKS**

(a) The graph of $y = f(x)$ is shown below.



Draw separate sketches of the following showing any critical features.

(i) $y = f(-x)$ 1

(ii) $y = \frac{1}{f(x)}$ 2

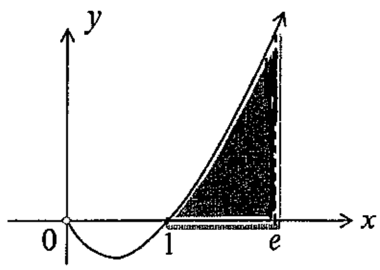
(iii) $y = f'(x)$ 2

(iv) $y = \sqrt{f(x)}$ 2

Question 3 is continued on page 4....

QUESTION 3 continued......... **MARKS**

(b) The shaded region in the diagram is bounded by the curve $y = x\ln x$, the x-axis and the line $x = e$. 4



The shaded region is rotated about the y-axis.

Use the method of cylindrical shells to find the volume of this solid.

(c) A particle of mass $0\cdot 1\,kg$ moving on a smooth horizontal table with constant speed $v\ ms^{-1}$ describes a circle with centre O and radius r metres.

The particle is attracted towards O by a force of magnitude $4v$ newtons and repelled from O by a force of magnitude $\frac{k}{r}$ newtons where k is a constant.

(i) Given that $v = 40$ and the time of one revolution is $\frac{\pi}{10}$ seconds, find the values of r and k. 2

(ii) If $r = 1$, find the set of possible values of k. 2

QUESTION 4 **BEGIN A NEW PAGE** **MARKS**

(a) (i) Prove that if a polynomial $P(x)$ has a root of multiplicity m then $P'(x)$ has this root of multiplicity $(m-1)$. 2

(ii) Find the value of k so that the equation $5x^5 - 3x^3 + k = 0$, has two equal roots, both positive. 2

(b) The roots of a cubic equation are α, β and γ.

They satisfy these equations

$$\alpha\beta\gamma = 10$$
$$\alpha^2\beta^2\gamma + \alpha^2\beta\gamma^2 + \alpha\beta^2\gamma^2 = 90$$
$$\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{2}{5}.$$

(i) Find the values of $\alpha + \beta + \gamma$ and $\alpha\beta + \alpha\gamma + \beta\gamma$. 2

(ii) Show that this cubic equation is $x^3 - 4x^2 + 9x - 10 = 0$. 1

(iii) Find the roots of this equation over the complex numbers. 2

(c) Let $z = \cos\theta + i\sin\theta$, where θ is real.

Consider the geometric series

$$S = 1 + \frac{iz}{3} - \frac{z^2}{9} - \frac{iz^3}{27} + \frac{z^4}{81} + \ldots\ldots$$

(i) Show that $S = \dfrac{3}{3 - iz}$. 1

(ii) Show that the imaginary part of S is $\dfrac{3\cos\theta}{10 + 6\sin\theta}$. 2

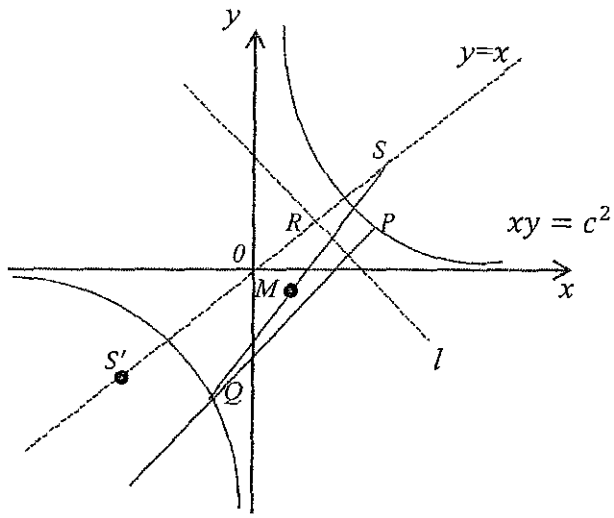
(iii) Find an expression for 3

$$1 - \frac{1}{3}\sin\theta - \frac{1}{9}\cos 2\theta + \frac{1}{27}\sin 3\theta + \frac{1}{81}\cos 4\theta + \ldots\ldots$$

in terms of $\sin\theta$.

QUESTION 5 **BEGIN A NEW PAGE** **MARKS**

(a)



$P(cp, \frac{c}{p})$ is a variable point on the hyperbola $xy = c^2$ such that $p > 0$.

$S(c\sqrt{2}, c\sqrt{2})$ is the focus of the hyperbola nearer to P, and the corresponding directrix l has equation $x + y = c\sqrt{2}$. The origin O is the centre of the hyperbola. The directrix l meets OS at R. The normal to the hyperbola at P cuts the hyperbola again at Q. M is the midpoint of QS.

(i) Show that $OS = 2c$ and R is the midpoint of OS. 2

(ii) Show that the normal at P has equation $p^2x - y = c(p^3 - \frac{1}{p})$. 2

(iii) Show that if the parameter at Q is q, then $qp^3 = -1$. 1

(iv) Show that as P varies, the coordinates of M satisfy $\left(x - \frac{c}{\sqrt{2}}\right)\left(y - \frac{c}{\sqrt{2}}\right) = \left(\frac{c}{2}\right)^2$. 2

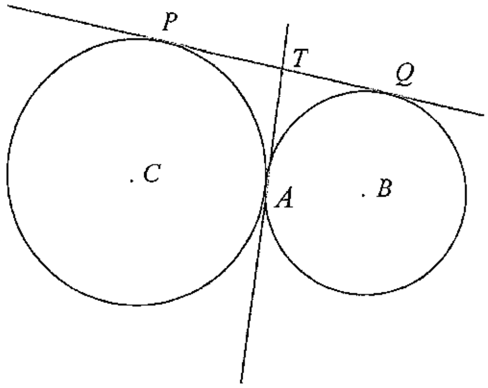
(v) Deduce that the locus of M is one branch of a hyperbola centred at R with foci O and S. 2

(vi) Write down the value of the eccentricity of the hyperbola which contains the locus of M. 1

Question 5 is continued on page 7....

QUESTION 5 continued........ **MARKS**

(b)



Two circles, centres C and B, touch externally at A. PQ is a direct common tangent touching the circles at P and Q respectively. The common tangent at A meets PQ at T.

(i) Show that the common tangent at A bisects PQ. 1

(ii) Let M be the midpoint of CB. Prove that $MT || CP$. 1

(iii) Prove that the circle with BC as diameter touches the line PQ. 3

QUESTION 6 **BEGIN A NEW PAGE** **MARKS**

(a) A plane curve is defined explicitly by the equation $x^2 + 2xy + y^5 = 4$. 3

This curve has a horizontal tangent at the point $P\ (x_1, y_1)$.

Show that x_1 must be a root of the equation $x^5 + x^2 + 4 = 0$.

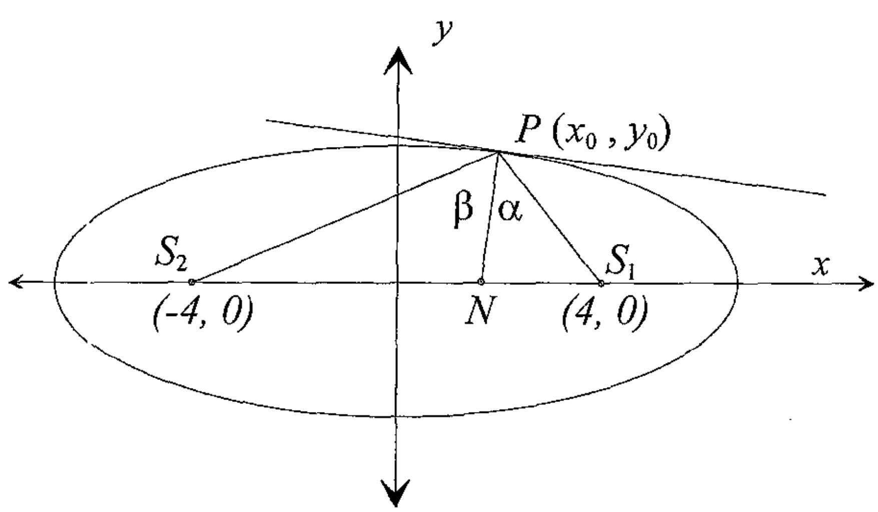
(b) (i) Show that the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the 2

point $P(x_0\ ,\ y_0)$ has equation $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$.

(ii) In the diagram below, the line PN is the normal to the ellipse 3

$\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ at $P(x_0\ ,\ y_0)$ and S_1 and S_2 are the foci of the ellipse.

$\angle NPS_1 = \alpha$ and $\angle NPS_2 = \beta$. Show that $\alpha = \beta$.



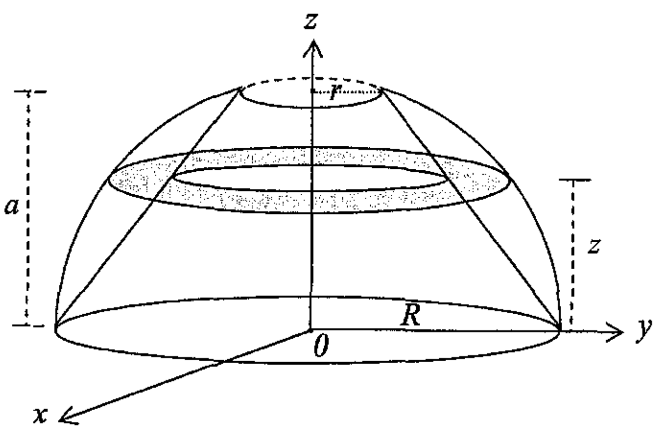
Question 6 is continued on page 9….

QUESTION 6 continued……… **MARKS**

(c) A hole in the shape of a truncated cone is cut through a hemisphere of radius R.

This hole has a radius r on the top and a radius R on the bottom. It is perpendicular to the xy plane and its axis of symmetry passes through the origin O, which is the centre of the hemisphere.

The cross section of the remaining solid S at a distance z from the xy plane is as shown in the diagram.



(i) Show that the area of the cross section of this solid is **3**

$$\pi[\,2RKz - (K^2 + 1)\,z^2\,], \text{ where } K = \frac{R-r}{a}.$$

(ii) Show that the volume of the solid S is **4**

$$V = \frac{a\pi R}{3}(R - r).$$

QUESTION 7 **BEGIN A NEW PAGE** **MARKS**

A particle of mass m kg is moving in a medium where the resistance is proportional to the speed. When the particle falls in this medium its terminal velocity is V ms^{-1}. The particle is projected vertically upwards with speed V, reaching a greatest height of H metres above its point of projection. The acceleration due to gravity is g ms^{-2}.

(i) If the resistance to motion has magnitude mkv, $k>0$, by considering forces acting on the particle, show that when it is falling $\ddot{x} = g - kv$. Hence express k in terms of V and g. **2**

(ii) By considering forces acting on the particle, show that when it is moving upwards $\ddot{x} = \frac{-g}{V}(V + v)$. **1**

(iii) By integration, show that $H = \frac{V^2}{g}(1 - ln2)$. **3**

(iv) Given that $\ddot{x} = \frac{g}{V}(V - v)$ when the particle is falling, show by integration that its speed v on its return to the projection point satisfies $\left(1 + \frac{v}{V}\right) + ln\left[\frac{1}{2}\left(1 - \frac{v}{V}\right)\right] = 0$. **4**

(v) Let $f(u) = (1 + u) + \ln\left[\frac{1}{2}(1 - u)\right]$. By considering the graphs of $y = -(1 + u)$ and $y = \ln\left[\frac{1}{2}(1 - u)\right]$, show that $f(u) = 0$ has a root u such that $0 < u < 1$. **2**

(vi) Using Newton's formula with a first approximation of $u_0 = 0 \cdot 6$, find a second approximation to $f(u) = 0,\ 0 < u < 1$. **2**

(vii) What approximate percentage of its terminal velocity has the particle acquired on its return journey to its point of projection? **1**

QUESTION 8 **BEGIN A NEW PAGE** **MARKS**

(a) (i) Show that $\cos n\theta - \cos(n-2)\theta = -2\sin\theta\sin(n-1)\theta$. 2

(ii) For each integer $n \geq 0$, let $I_n = \int \cos n\theta \operatorname{cosec}\theta \, d\theta$. 2

Show that $I_n - I_{n-2} = \frac{2}{n-1}\cos(n-1)\theta + c$,

where c is a constant and $n \neq 1$.

(iii) Hence, evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 5\theta \operatorname{cosec}\theta \, d\theta$. 3

(b) Let $f(x) = \ln x - x + 1$, where $x > 0$

(i) Show that $\ln x \leq x - 1$. 2

(ii) Let $p = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$, where $a_1, a_2, a_3, \ldots, a_n$ are positive real numbers. 2

Show that $\ln\left(\frac{a_1\, a_2\, a_3 \ldots a_n}{p^n}\right) \leq \frac{a_1 + a_2 + \ldots + a_n}{p} - n$.

(iii) Show that $\sqrt[n]{a_1\, a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}$. 2

(iv) Hence or otherwise, show that $\frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_4} + \ldots + \frac{x_n}{x_1} \geq n$. 2

THE END

CTHS

QUESTION 1 2011 EXT 2 SOLNS

(a) $\int \frac{\sec^2 x}{\sqrt{1+2\tan x}}\,dx$

Let $u = 1 + 2\tan x$

$du = 2\sec^2 x\,dx$

$\int \frac{\sec^2 x}{\sqrt{1+2\tan x}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= u^{1/2} + c$

$= \sqrt{1+2\tan x} + c$

(b) $\int_1^{\sqrt{3}} \frac{1}{(1+x^2)\tan^{-1}x}\,dx = \int_{\pi/4}^{\pi/3} \frac{1}{u}\,du$

$= \left[\ln u\right]_{\pi/4}^{\pi/3}$

$= \ln \frac{\pi}{3} - \ln \frac{\pi}{4}$

$= \ln\left(\frac{\pi}{3} \div \frac{\pi}{4}\right)$

$= \ln \frac{4}{3}$

(c) $\int \tan^{-1}x = \int \tan^{-1}x \cdot 1\,dx$

$= x\tan^{-1}x - \int x \cdot \frac{1}{1+x^2}\,dx$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

$= x\tan^{-1}x - \ln\sqrt{1+x^2} + c$

(d)(i) $\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$

$= \frac{1}{2}(1+\tan^2\frac{\theta}{2})$

$= \frac{1}{2}(1+t^2)$

$\therefore d\theta = \frac{2}{1+t^2}\,d\theta$

(ii) $t = \tan\frac{\theta}{2}$

$\sin\theta = \frac{2t}{1+t^2}$

$\mathrm{cosec}\,\theta = \frac{1+t^2}{2t}$

$\theta = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$

$\theta = \frac{\pi}{3}$; $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\int_{\pi/6}^{\pi/2} 2\,\mathrm{cosec}^2\theta\tan\frac{\theta}{2}\,d\theta = \int_{\frac{1}{\sqrt{3}}}^{1} 2\left(\frac{1+t^2}{2t}\right)^2 \cdot t \cdot \frac{2}{1+t^2}$

$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1+t^2}{t}\,dt$

$= \int_{\frac{1}{\sqrt{3}}}^{1} \left(\frac{1}{t} + t\right)dt$

$= \left[\ln t + \frac{t^2}{2}\right]_{\frac{1}{\sqrt{3}}}^{1}$

$= \left(\ln 1 + \frac{1}{2}\right) - \left(\ln\frac{1}{\sqrt{3}} + \frac{1}{6}\right)$

$= \frac{1}{3} + \frac{1}{2}\ln 3$

e) $\int \frac{x+4}{(x-1)(x^2+4)}\,dx$

$\frac{x+4}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

$x+4 = A(x^2+4) + (Bx+C)(x-1)$

put $x = 1$

$1+4 = A(1+4) + 0$

$A = 1$

equate coefficients of x^2

$0 = A + B$

$\therefore B = -1$

equate constant terms

$4 = 4A - C$

$4 = 4\times 1 - C$

$C = 0$

$\int \frac{x+4}{(x-1)(x^2+4)}\,dx = \int \left(\frac{1}{x-1} + \frac{-x}{x^2+4}\right)dx$

$= \ln(x-1) - \frac{1}{2}\ln(x^2+4) + C$

or $\ln\frac{x-1}{\sqrt{x^2+4}} + C$

QUESTION 2

(a)(i) $\frac{1}{z} = \frac{1}{1-2i} \times \frac{1+2i}{1+2i}$

$= \frac{1}{5} + \frac{2}{5}i$

(ii) $\bar{z}(z - \bar{z}) = (1+2i)(1-2i-(1+2i))$

$= (1+2i) \cdot -4i$

$= 8 - 4i$

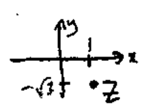
(iii) $\frac{z}{i} + i\bar{z} = \frac{1-2i}{i} + i \cdot (1+2i)$

$= -2 - i + i - 2$

$= -4$

(b)(i) $r = \sqrt{1+3}$

$= 2$



$\tan\theta = -\sqrt{3}$

$\theta = -\frac{\pi}{3}$

$z = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

(ii) $z^6 = 2^6\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]^6$

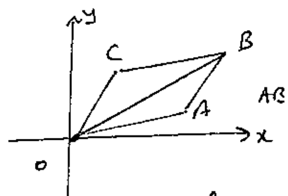
$= 64\,\mathrm{cis}\left(-\frac{6\pi}{3}\right)$

$= 64$

which is an integer

Q2 continued

(c)



$OA = |z_1|$

$AB = OC = |z_2|$

$OB = |z_1 + z_2|$

Sum of any 2 sides of a Δ is greater than third side

$OA + AB > OB$

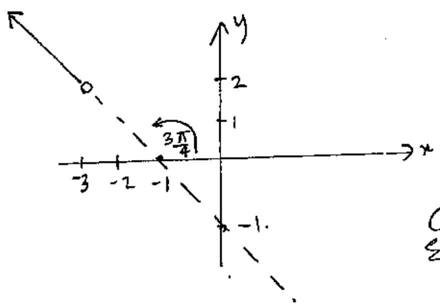
$|z_1| + |z_2| > |z_1 + z_2|$

$\text{Arg } z_1 \neq \text{Arg } z_2$

(d) $\text{Arg}(z + 3 - 2i) = \frac{3\pi}{4}$

$\text{Arg}(z - (-3 + 2i)) = \frac{3\pi}{4}$

$\tan \frac{3\pi}{4} = -1$



Cartesian Equation is $y = -x - 1$, $x < -3$

(e)(i) Complex number represented by R is addition of complex numbers represented by P and Q

$\vec{OP} = \vec{QR}$

$\vec{OQ} = \vec{PR}$

∴ OPRQ is a parallelogram

OQ is obtained by multiplying OP by $i\sqrt{3}$. multiplying by i corresponds to a rotation of 90° in an anticlockwise direction.

∴ $\angle POQ = 90°$

∴ OPRQ is a rectangle

(ii) $|z + i\sqrt{3}z| = \sqrt{(OP)^2 + (RP)^2}$

$OP = |z| = 1$

$RP = OQ = |i\sqrt{3}z| = |i|\,|\sqrt{3}|\,|z| = \sqrt{3}$

$\therefore OR = |z + i\sqrt{3}z| = \sqrt{1+3} = 2$

(iii) $\text{Arg}(z + i\sqrt{3}z) = \text{Arg}[z(1 + i\sqrt{3})]$

$= \text{Arg } z + \text{Arg}(1 + i\sqrt{3})$

$\text{Arg } z = \alpha$

$\text{Arg}(1 + i\sqrt{3}) = \pi/3$

$\therefore \text{Arg}(z + i\sqrt{3}z) = \alpha + \pi/3$

(iv) $\text{Im}(z + i\sqrt{3}z) = \text{Im}\left[2\left(\cos(\alpha + \frac{\pi}{3}) + i\sin(\alpha + \frac{\pi}{3})\right)\right]$

$= 2\sin(\alpha + \frac{\pi}{3})$ ——— (1)

also $\text{Im}(z + i\sqrt{3}z) = \text{Im } z_i + \text{Im}(i\sqrt{3}z)$

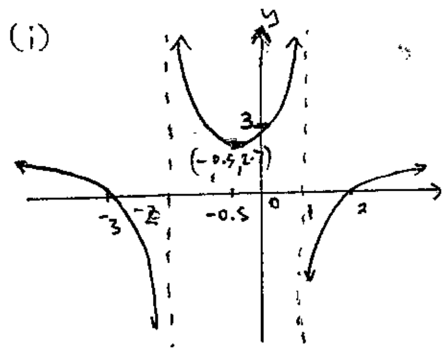
$= \sin\alpha + \text{Im}(i\sqrt{3}\cos\alpha - \sqrt{3}\sin\alpha)$

$= \sin\alpha + \sqrt{3}\cos\alpha$ ——— (2)

from (1) and (2) $\sin\alpha + \sqrt{3}\cos\alpha = 2\sin(\alpha + \frac{\pi}{3})$

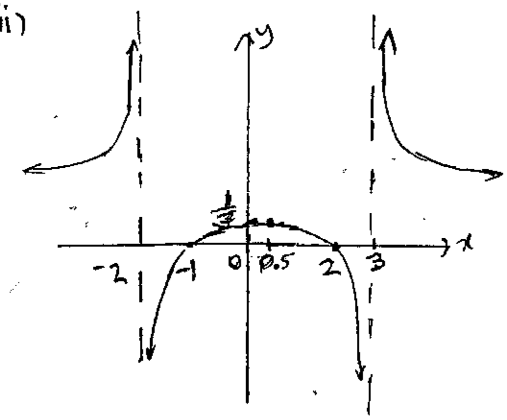
QUESTION 3

(a) (i)

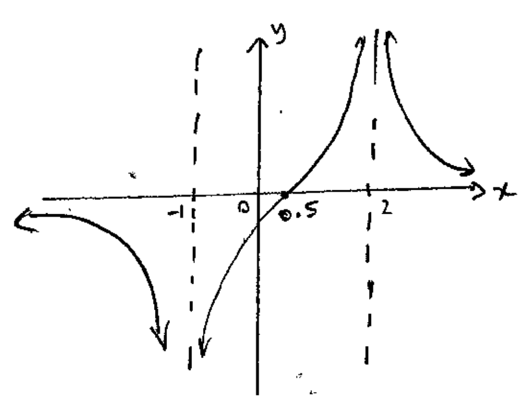


$y = f(-x)$

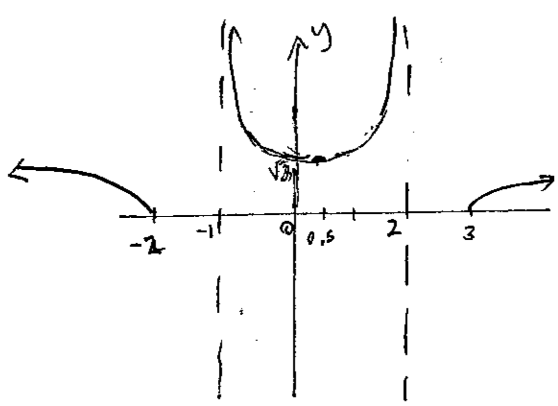
(ii)



$y = \frac{1}{f(x)}$



$y = f'(x)$



$y = \sqrt{f(x)}$

Q3b) volume cylindrical shell $= 2\pi r h \,\delta x$

$= 2\pi\, x . x\ln x \,\delta x$

$$V = 2\pi \int_1^e x^2 \ln x \, dx$$

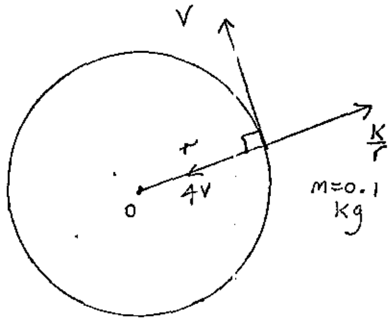
$$= 2\pi\left[\frac{x^3}{3}\ln x\right]_1^e - 2\pi\int_1^e \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$$= 2\pi\left[\frac{e^3}{3}\ln e - \frac{1}{3}\ln 1\right] - 2\pi\left[\frac{x^3}{9}\right]_1^e$$

$$= \frac{2\pi e^3}{3} - 2\pi\left[\frac{e^3}{9} - \frac{1}{9}\right]$$

$$= \frac{2\pi}{9}\left[2e^3 + 1\right] \text{ units}^3$$

(c)



(i) $T = \frac{\pi}{10}$

$\therefore \omega = \frac{2\pi}{\pi/10} = 20$ rad/s

$v = 40$

$v = r\omega$

$40 = 20r$

$r = 2$

$4v - \frac{K}{r} = \frac{mv^2}{r}$ ——— ①

$v = 40 \quad r = 2$

$4\times 40 - \frac{K}{2} = \frac{0.1\times 40^2}{2}$

$160 - \frac{K}{2} = 80$

$K = 160$

(ii) sub $r = 1$ into equ ①

$4v - K = mv^2$

$0.1v^2 = 4v - K$

$v^2 - 40v + 10K = 0$

for motion to continue as described $\Delta \geq 0$

$\therefore (-40)^2 - 4\times 1\times 10K \geq 0$

$40(40 - K) \geq 0$

$40 - K \geq 0$

$\therefore K \leq 40$

but $K \geq 0$ for motion to continue as described

$\therefore 0 \leq K \leq 40$

QUESTION 4

a)(i) Let root of multiplicity m be α

$\therefore P(x) = (x-\alpha)^m . Q(x)$

$P'(x) = m(x-\alpha)^{m-1} . Q(x) + (x-\alpha)^m . Q'(x)$

$= (x-\alpha)^{m-1}\left[m.Q(x) + Q'(x)(x-\alpha)\right]$

$= (x-\alpha)^{m-1} . S(x)$.

$\therefore P'(x)$ has α as a root of multiplicity $(m-1)$ and $P(x)$ has α as a root of multiplicity m.

(ii) $f(x) = 5x^5 - 3x^3 + K$

$f(x) = 0$ has 2 equal positive roots

$\Rightarrow f'(x) = 0$ has same root as a single root

$f'(x) = 25x^4 - 9x^2$

$f'(x) = 0$

$25x^4 - 9x^2 = 0$

$x^2(25x^2 - 9) = 0$

$x = 0,\ x = \pm\frac{3}{5}$

take $x = \frac{3}{5}$ as the equal roots are positive

$\Rightarrow f\left(\frac{3}{5}\right) = 0$

$5\left(\frac{3}{5}\right)^5 - 3\left(\frac{3}{5}\right)^3 + K = 0$

$K = \frac{81}{125} - \frac{243}{625}$

$= \frac{162}{625}$

b)

i) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} = \frac{2}{5}$

$\therefore \frac{\gamma+\beta+\alpha}{\alpha\beta\gamma} = \frac{2}{5}$

$\frac{\gamma+\beta+\alpha}{10} = \frac{2}{5}$

$\therefore \alpha+\beta+\gamma = 4$ ——— ①

$\alpha^2\beta^2\gamma + \alpha^2\beta\gamma^2 + \alpha\beta^2\gamma^2 = 90$

$\alpha\beta\gamma(\alpha\beta + \alpha\gamma + \beta\gamma) = 90$

$\therefore 10(\alpha\beta + \alpha\gamma + \beta\gamma) = 90$

$\alpha\beta + \alpha\gamma + \beta\gamma = 9$

(ii) cubic equation has form

$x^3 - (\Sigma\alpha)x^2 + (\Sigma\alpha\beta)x - (\Sigma\alpha\beta\gamma) = 0$

$\therefore x^3 - 4x^2 + 9x - 10 = 0$

b)(iii)

$f(2) = 2^3 - 4(2)^2 + 9(2) - 10 = 0$

$\therefore x = 2$ is a root

Let the roots be $\alpha, \beta, 2$.

from ① $\Rightarrow \alpha + \beta = 2$

Since $\alpha\beta\gamma = 10$, $\alpha\beta = 5$

$\therefore$ quadratic equation with roots α, β is

$x^2 - 2x + 5 = 0$

$x = \dfrac{2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2}$

$= \dfrac{2 \pm \sqrt{-16}}{2}$

$= \dfrac{2 \pm 4i}{2}$

$= 1 \pm 2i$

$\therefore$ The roots of the quadratic equation are

$x = 2, 1-2i, 1+2i$

(c)(i) series S is an infinite geometric series

$a = 1$, $r = \dfrac{iz}{3}$

$S_\infty = \dfrac{1}{1 - \frac{iz}{3}}$

$= \dfrac{3}{3 - iz}$

(ii) $\dfrac{3}{3 - iz} = \dfrac{3}{3 - i(\cos\theta + i\sin\theta)}$

$= \dfrac{3}{3 - i\cos\theta + \sin\theta}$

$= \dfrac{3}{(3+\sin\theta) - i\cos\theta} \times \dfrac{(3+\sin\theta) + i\cos\theta}{(3+\sin\theta) + i\cos\theta}$

$= \dfrac{3(3+\sin\theta) + 3i\cos\theta}{(3+\sin\theta)^2 + \cos^2\theta}$ —— (*)

Imaginary part of S is

$\dfrac{3\cos\theta}{(3+\sin\theta)^2 + \cos^2\theta} = \dfrac{3\cos\theta}{9 + 6\sin\theta + \sin^2\theta + \cos^2\theta}$

$= \dfrac{3\cos\theta}{10 + 6\sin\theta}$

(iii) $S = 1 + \dfrac{iz}{3} - \dfrac{z^2}{9} - \dfrac{iz^3}{27} + \dfrac{iz^4}{81} + \dots$

$= 1 + \dfrac{i(\cos\theta + i\sin\theta)}{3} - \dfrac{(\cos\theta + i\sin\theta)^2}{9} - i\dfrac{(\cos\theta + i\sin\theta)^3}{27} + \dfrac{(\cos\theta + i\sin\theta)^4}{81} + \dots$

$= 1 + \dfrac{i\cos\theta - \sin\theta}{3} - \dfrac{\cos 2\theta + i\sin 2\theta}{9} - \dfrac{i\cos 3\theta - \sin 3\theta}{27} + \dfrac{\cos 4\theta + i\sin 4\theta}{81} + \dots$

$Re(s) = 1 - \dfrac{\sin\theta}{3} - \dfrac{\cos 2\theta}{9} + \dfrac{\sin 3\theta}{27} + \dfrac{\cos 4\theta}{81} + \dots$ —— (1)

from *

$Re(s) = \dfrac{3(3+\sin\theta)}{(3+\sin\theta)^2 + \cos^2\theta} = \dfrac{3(3+\sin\theta)}{10 + 6\sin\theta}$ —— (2)

equating (1), (2) gives $1 - \frac{1}{3}\sin\theta - \frac{1}{9}\cos 2\theta + \frac{1}{27}\sin 3\theta + \frac{1}{81}\cos 4\theta + \dots = \dfrac{3(3+\sin\theta)}{10 + 6\sin\theta}$

QUESTION 5

i) $OS = \sqrt{2c^2+2c^2}$

$= \sqrt{4c^2}$

$= 2c$

R is intersection of $y = x$ —— (1)

and $x+y = c\sqrt{2}$ —— (2)

solve simultaneously

$x + x = c\sqrt{2}$

$x = \frac{c\sqrt{2}}{2}$

$y = \frac{c\sqrt{2}}{2}$

$\therefore$ R is $\left(\frac{c\sqrt{2}}{2}, \frac{c\sqrt{2}}{2}\right)$

midpoint of OS is $\left(\frac{0+c\sqrt{2}}{2}, \frac{0+c\sqrt{2}}{2}\right)$

$= \left(\frac{c\sqrt{2}}{2}, \frac{c\sqrt{2}}{2}\right)$

$\therefore$ R is midpt of OS.

ii) $xy = c^2 \Rightarrow x = ct,\ y = \frac{c}{t}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= -\frac{c}{t^2} \cdot \frac{1}{c}$

$= -\frac{1}{t^2}$

at $P(cp, \frac{c}{p})$ $\frac{dy}{dx} = -\frac{1}{p^2}$

$\therefore$ slope normal $= p^2$

equation normal is

$y - \frac{c}{p} = p^2(x - cp)$

$p^2x - y = cp^3 - \frac{c}{p}$

$p^2x - y = c\left(p^3 - \frac{1}{p}\right)$

iii) $Q(cq, \frac{c}{q})$ lies on normal

$\therefore p^2cq - \frac{c}{q} = c\left(p^3 - \frac{1}{p}\right)$

$p^2cq - cp^3 = \frac{c}{q} - \frac{c}{p}$

$cp^2(q-p) = c\left(\frac{p-q}{pq}\right)$

$\div(q-p)$ as $p \neq q$ →

$p^2 = -\frac{1}{pq}$

$\therefore p^3q = -1$

Q5 (a)

(iv) $S(c\sqrt{2}, c\sqrt{2})$

$Q\left(-\frac{c}{p^3}, -cp^3\right)$

as $q = -\frac{1}{p^3}$ from (iii)

$\therefore$ M is $\left(\frac{c\sqrt{2} - \frac{c}{p^3}}{2}, \frac{c\sqrt{2} - cp^3}{2}\right)$ —— (*) or see below

$= \left(\frac{c}{2}\left(\sqrt{2} - \frac{1}{p^3}\right), \frac{c}{2}\left(\sqrt{2} - p^3\right)\right)$

Show M satisfies $\left(x - \frac{c}{\sqrt{2}}\right)\left(y - \frac{c}{\sqrt{2}}\right) = \left(\frac{c}{2}\right)^2$

$LHS = \left(\frac{c}{2}\left(\sqrt{2} - \frac{1}{p^3}\right) - \frac{c}{\sqrt{2}}\right)\left(\frac{c}{2}\left(\sqrt{2} - p^3\right) - \frac{c}{\sqrt{2}}\right)$

$= -\frac{c}{2p^3} \times -\frac{p^3c}{2}$

$= \frac{c^2}{4}$

$= \left(\frac{c}{2}\right)^2$

$= RHS$

* $\therefore$ M satisfies $\left(x - \frac{c}{\sqrt{2}}\right)\left(y - \frac{c}{\sqrt{2}}\right) = \left(\frac{c}{2}\right)^2$

OR use $M = \left(\frac{c\sqrt{2} + cq}{2}, \frac{c\sqrt{2} + \frac{c}{q}}{2}\right)$ and show M satisfies

(v)

$\left(x - \frac{c}{\sqrt{2}}\right)\left(y - \frac{c}{\sqrt{2}}\right) = \left(\frac{c}{2}\right)^2$ is the equation of a rectangular hyperbola centred at R with foci on $y = x$.

$OR = SR = \frac{1}{2}OS = c = 2\left(\frac{c}{2}\right)$ and so O and S are the foci of the rectangular hyperbola.

Consider coordinates of M in terms of p.

Using (*) if $p > 0$ then $x < \frac{c}{\sqrt{2}}$ and $y < \frac{c}{\sqrt{2}}$ which restricts the locus of M the branch of the rectangular hyperbola with centre $\left(\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$ which is closer to O.

vi) As hyperbola is rectangular eccentricity $= \sqrt{2}$

Q5(b)(i) For circle centre C

PT = TA (tangents drawn from an external point are equal)

For circle centre B

TA = TQ (tangents drawn from an external point are equal)

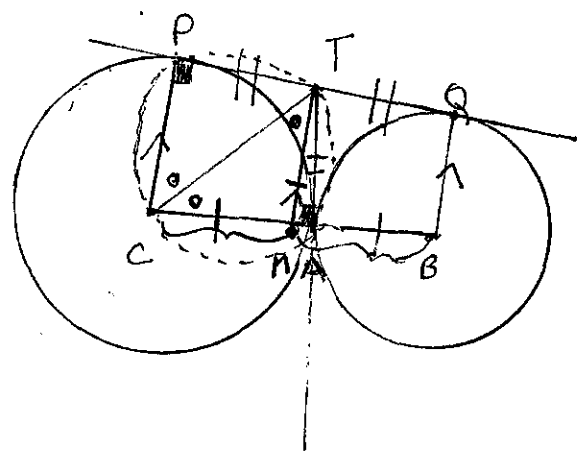
∴ PT = TQ

∴ The common tangent at A bisects PQ

(ii) MT bisects CB and PQ (PT = TQ (proved), M is midpoint of CB)

MT ∥ CP ∥ QB (equal intercepts cut on both transversals CB and PQ)

(iii)



<CPT = <TAC (tangent ⊥ radius at point of contact)
= 90°
and <CPT + <TAC = 180°
∴ PTAC is a cyclic quadrilateral (opposite angles cyclic quadrilateral are supplementary)

< PCT = <TCA (angles at the circumference standing on equal chords (or arcs) are equal)

<PCT = <CTA (alternate angles are equal PC ∥ TM)

∴ <TCA = <CTA
and CM = TM (In any Δ sides opposite = <'s are =)
= MB
∴ BC is diameter of a circle touching PQ

QUESTION 6

(a) $x^2+2xy+y^5=4$ ——— ①

$2x+2x\cdot\frac{dy}{dx}+y\cdot 2+5y^4\frac{dy}{dx}=0$

$\frac{dy}{dx}(2x+5y^4)=-2x-2y$

$\frac{dy}{dx}=\frac{-2x-2y}{2x+5y^4}$

for horizontal tangent $\frac{dy}{dx}=0$ and $x=x_1$, $y=y_1$

$\frac{-2x_1-2y_1}{2x_1+5y_1^4}=0$

$-2x_1-2y_1=0$

$\therefore y_1=-x_1$

Sub into equation ①

$(x_1)^2+2(x_1)(-x_1)+(-x_1)^5=4$

$x_1^2-2x_1^2-x_1^5-4=0$

$x_1^5+x_1^2+4=0$

$\therefore x_1$ is a root of $x^5+x^2+4=0$

(b)(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-\frac{2x}{a^2}}{\frac{2y}{b^2}}$

$=-\frac{b^2x}{a^2y}$

at (x_0,y_0) slope of tangent $=\frac{-b^2x_0}{a^2y_0}$

equation of tangent is $y-y_0=\frac{-b^2x_0}{a^2y_0}(x-x_0)$

$a^2y_0y-a^2y_0^2=-b^2x_0x+b^2x_0^2$

$(\div a^2b^2)$ $\frac{y_0y}{b^2}-\frac{y_0^2}{b^2}=-\frac{x_0x}{a^2}+\frac{x_0^2}{a^2}$

$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}$ but $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$

$\therefore \frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$

P11

6 (b)(ii) $\frac{x^2}{25}+\frac{y^2}{9}=1$

using (b)(i) and $a^2=25$, $b^2=9$

$\frac{dy}{dx}=\frac{-9x}{25y}$

(x_0,y_0) slope tangent $=\frac{-9x_0}{25y_0}$

slope normal $=\frac{25y_0}{9x_0}$ $(m_1m_2=-1)$

gradient $PS_1=\frac{y_0}{x_0-4}$

gradient $PS_2=\frac{y_0}{x_0+4}$

$\tan\alpha=\left|\frac{\frac{25y_0}{9x_0}-\frac{y_0}{x_0-4}}{1+\frac{25y_0}{9x_0}\cdot\frac{y_0}{x_0-4}}\right|$

$=\left|\frac{25x_0y_0-100y_0-9x_0y_0}{9x_0^2-36x_0+25y_0^2}\right|$

$=\left|\frac{4y_0(4x_0-25)}{225-36x_0}\right|$ since P lies on $9x_0^2+25y_0^2=225$

$=\left|\frac{4y_0(4x_0-25)}{9(25-4x_0)}\right|$

$=\left|\frac{4y_0}{9}\right|$

$\tan\beta=\left|\frac{\frac{25y_0}{9x_0}-\frac{y_0}{x_0+4}}{1+\frac{25y_0}{9x_0}\cdot\frac{y_0}{x_0+4}}\right|$

$=\left|\frac{25x_0y_0+100y_0-9x_0y_0}{9x_0^2+36x_0+25y_0^2}\right|$

$=\left|\frac{4y_0(4x_0+25)}{225+36x_0}\right|$ since P lies on $9x_0^2+25y_0^2=225$

$=\left|\frac{4y_0(4x_0+25)}{9(25+4x_0)}\right|$

$=\left|\frac{4y_0}{9}\right|$

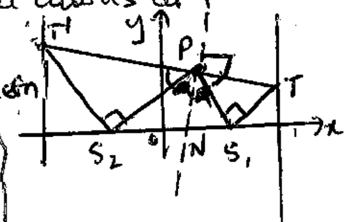
$=\tan\alpha$

$\therefore \alpha=\beta$

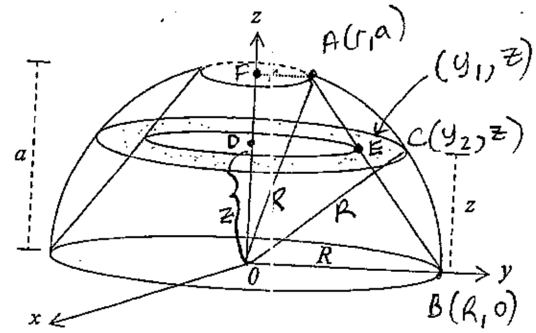
P12

or use "segment of the tangent to an ellipse between point of contact and directrix subtends a right angle at corresponding focus" i.e. "tangent (and hence normal) at P on ellipse is equally inclined to focal chords at P" → use $m_1m_2=-1$

use similarity and focus-directrix definition of conic



6(c)



(i)
Area of cross section is annulus

$A = \pi(y_2^2 - y_1^2)$

In Δ OCD $\quad z^2 + y_2^2 = R^2 \quad \therefore y_2^2 = R^2 - z^2$ —— (1)

$M_{AB} = \frac{a}{r-R}$

Equation AB $\quad z - 0 = \frac{a}{r-R}(y - R)$

$y = \frac{(r-R)}{a} z + R$

as E lies on AB then

$y_1 = \frac{r-R}{a} z + R$

} or use similar Δ's

$K = \frac{R-r}{a}$

$\therefore y_1 = -Kz + R$

$y_1^2 = (R - Kz)^2$ —— (2)

using (1) & (2)

$\therefore A = \pi\left[(R^2 - z^2) - (R - Kz)^2\right]$

$= \pi\left[R^2 - z^2 - R^2 + 2KRz - K^2z^2\right]$

$= \pi\left[-z^2 + 2KRz - K^2z^2\right]$

$= \pi\left[2KRz - (K^2+1)z^2\right]$ as required

(ii) $V = \lim_{\delta z \to 0} \sum_{z=0}^{a} \pi\left[2KRz - (K^2+1)z^2\right]\delta z$

$= \pi\int_0^a (2KRz - (K^2+1)z^2)\,dz$

$= \pi\left[KRz^2 - (K^2+1)\frac{z^3}{3}\right]_0^a$

$= \pi\left[KRa^2 - (K^2+1)\frac{a^3}{3} - 0\right]$

$= \frac{a\pi}{3}\left(3KRa - K^2a^2 - a^2\right)$

$= \frac{a\pi}{3}\left[3Ra\left(\frac{R-r}{a}\right) - a^2\left(\frac{R-r}{a}\right)^2 - a^2\right]$

$= \frac{a\pi}{3}\left[3R^2 - 3Rr - (R^2 - 2Rr + r^2) - a^2\right]$

$= \frac{a\pi}{3}\left[2R^2 - Rr - r^2 - a^2\right]$

In ΔOAF $\quad r^2 + a^2 = R^2$

$\therefore V = \frac{a\pi}{3}(2R^2 - Rr - R^2)$

$= \frac{a\pi R}{3}(R - r)$ as required

QUESTION 7

(i) $x\downarrow$ $\uparrow mkV$, $\downarrow mg$

$m\ddot{x} = mg - mkV$

$\therefore \ddot{x} = g - kV$

$\ddot{x} \to 0$

$g - kV \to 0$

so $V \to \frac{g}{k}$

terminal velocity is $\mathcal{V}$ $\Rightarrow \mathcal{V} = \frac{g}{k}$

$\therefore k = \frac{g}{\mathcal{V}}$

(ii) $x\uparrow$ $\downarrow mkv$, $\downarrow mg$

$m\ddot{x} = -mkv - mg$

$\ddot{x} = -kv - g$

$= -k\left[\frac{g}{k} + v\right]$

$= -\frac{g}{\mathcal{V}}\left[\mathcal{V} + v\right]$

(iii) $v\frac{dv}{dx} = \frac{-g}{\mathcal{V}}\left[\mathcal{V} + v\right]$

$\frac{v}{\mathcal{V}+v}\,dv = -\frac{g}{\mathcal{V}}\,dx$

$1 - \mathcal{V}\frac{1}{\mathcal{V}+v}\,dv = -\frac{g}{\mathcal{V}}\,dx$

$\int_{\mathcal{V}}^{0}\left(1 - \mathcal{V}\frac{1}{\mathcal{V}+v}\right)dv = \int_0^H -\frac{g}{\mathcal{V}}\,dx$

$\left[v - \mathcal{V}\ln(v + \mathcal{V})\right]_{\mathcal{V}}^{0} = \left[-\frac{g}{\mathcal{V}}x\right]_0^H$

$(-\mathcal{V}\ln\mathcal{V}) - (\mathcal{V} - \mathcal{V}\ln 2\mathcal{V}) = -\frac{g}{\mathcal{V}}H$

$H = -\frac{\mathcal{V}}{g}\left[-\mathcal{V}\ln\mathcal{V} - \mathcal{V} + \mathcal{V}\ln 2\mathcal{V}\right]$

$= \frac{\mathcal{V}^2}{g}\left[\ln\mathcal{V} + 1 - \ln 2\mathcal{V}\right]$

$= \frac{\mathcal{V}^2}{g}\left[1 + \ln\frac{\mathcal{V}}{2\mathcal{V}}\right]$

$= \frac{\mathcal{V}^2}{g}\left[1 - \ln 2\right]$ $\quad$ *

(iv) $\ddot{x} = \frac{g}{\mathcal{V}}(\mathcal{V} - v)$ $\quad$ initial conditions $t = 0$, $x = 0$, $v = 0$ (restart of downwards motion)

$v\frac{dv}{dx} = \frac{g}{\mathcal{V}}\left[\mathcal{V} - v\right]$

$\frac{dv}{dx} = \frac{g}{\mathcal{V}}\left[\frac{\mathcal{V} - v}{v}\right]$

$dx = \frac{\mathcal{V}}{g}\left[\frac{v}{\mathcal{V} - v}\right]dv$

$= \frac{-\mathcal{V}}{g}\left[\frac{\mathcal{V} - v - \mathcal{V}}{\mathcal{V} - v}\right]dv$

$$dx = -\frac{V}{g}\left[1 - \frac{V}{V-v}\right]dv$$

$$x = -\frac{V}{g}\left[v + V\ln|V-v|\right] + c$$

$x=0$, $v=0$

$$0 = -\frac{V}{g}\left[V\ln V\right] + c$$

$$c = \frac{V}{g}\left[V\ln V\right]$$

$$\therefore x = -\frac{V}{g}\left[v + V\ln|V-v|\right] + \frac{V}{g}\left[V\ln V\right]$$

$$= -\frac{V}{g}\left[v + V\ln\left|\frac{V-v}{V}\right|\right]$$

(iv) alternate from ∗

$$\frac{v}{V-v}\,dv = \frac{g}{V}\,dx$$

'V' as we need to find it

$$\int_0^V \frac{v}{V-v}\,dv = \int_0^H \frac{g}{V}\,dx$$

$$\int_0^V \left[-1 + \frac{V}{V-v}\right]dv = \int_0^H \frac{g}{V}\,dx$$

$$\left[-v - V\ln|V-v|\right]_0^V = \frac{g}{V}H$$

$$H = -\frac{V}{g}\left[v + V\ln|V-v| - (0 + V\ln V)\right]$$

$$= -\frac{V}{g}\left[v + V\ln\left|\frac{V-v}{V}\right|\right]$$

Distance is H from ∗

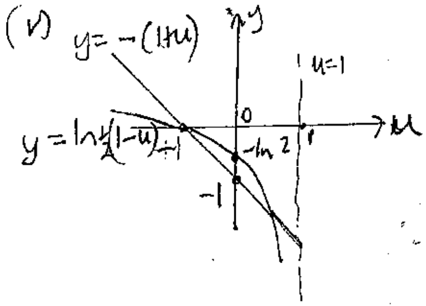
$$\frac{V^2}{g}\left[1 - \ln 2\right] = -\frac{V}{g}\left[v + V\ln\left|\frac{V-v}{V}\right|\right]$$

$\left(\times\frac{g}{V}\right)$ $\quad V - V\ln 2 + v + V\ln\left(1 - \frac{v}{V}\right) = 0$

$$V + v + V\ln\tfrac{1}{2}\left(1 - \frac{v}{V}\right) = 0$$

$(\div V)$ $\quad 1 + \frac{v}{V} + \ln\frac{1}{2}\left(1 - \frac{v}{V}\right) = 0$ as required

(v)



$f(u) = (1+u) + \ln\left[\frac{1}{2}(1-u)\right]$

at intersection of $y = -(1+u)$ and $y = \ln\frac{1}{2}(1-u)$, $f(u) = 0$

from graph, only one root u as only one intersection point for $0 < u < 1$

(vi) $f'(u) = 1 - \frac{1}{1-u}$

$$= \frac{-u}{1-u}$$

$$u_1 = 0.6 - \frac{f(0.6)}{f'(0.6)}$$

$$= 0.6 - \frac{-0.0094379}{-1.5}$$

$$= 0.59$$

(vii) The particle has acquired $\doteqdot 60\%$ of its terminal velocity on return to projection point